

Quantum Brans-Dicke Gravity in Euclidean Path Integral Formulation

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Abstract

The conformal structure of Brans-Dicke gravity action is carefully studied. It is discussed that Brans-Dicke gravity action has definitely no conformal invariance. It is shown, however, that this lack of conformal invariance enables us to demonstrate that Brans-Dicke theory appears to have a better short-distance behavior than Einstein gravity as far as Euclidean path integral formulation for quantum gravity is concerned.

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I. Introduction

In general, at the classical level (ignoring the quantum effects such as the conformal anomaly that could break the conformal invariance explicitly), one can always construct a conformally-invariant (Weyl-invariant) matter field action. It is not clear, however, whether or not the gravity sector of the action also needs to be conformally-invariant. For instance, if one considers Einstein gravity, it seems to suggest that the gravity action, in general, is not necessarily conformally-invariant since the Einstein-Hilbert action is manifestly Weyl non-invariant. Here in the present work, we are interested in the conformal nature of Brans-Dicke (BD) theory of gravity¹. Since its conformal property has hardly been studied carefully², we would like to take a closer look at its conformal structure in this work. We shall show that BD gravity action has no conformal invariance. Remarkably, however, it will be shown that this lack of conformal invariance enables us to demonstrate that the BD gravity theory possesses a much better short-distance behavior than Einstein gravity. The BD gravity action in the *Euclidean* signature takes the form

$$I_E[g, \Phi] = -\frac{1}{16\pi} \left[\int_M d^4x \sqrt{g} \{ \Phi(R - 2\Lambda) - \omega g^{\mu\nu} \frac{(\partial_\mu \Phi)(\partial_\nu \Phi)}{\Phi} \} + 2 \int_{\partial M} d^3x \sqrt{h} \Phi (K - K_0) \right] \quad (1)$$

where the “BD ω -parameter” is positive by definition and K is the trace of the second fundamental form of the boundary ∂M and $h_{\mu\nu}$ is the 3-metric induced on ∂M . K_0 is the trace of the second fundamental form of ∂M embedded in flat spacetime. This constant term involving K_0 has been added so that in cases when ∂M is embedded in flat spacetime the action of flat spacetime can be normalized to zero.

II. Conformally Invariant ?

Upon examining the structure of BD gravity action, (but without the cosmological constant, which is the potential term that explicitly breaks the classical conformal invariance) one would naturally be tempted to the possibility of conformal (Weyl) invariance and demand it since the BD gravity action happens to have no dimensionful coupling constants at all which is the “bottom-line qualification” for any field theory action to be scale invariant. In other words, in BD gravity action, the dimensionful Newton’s constant is replaced by

the “BD-scalar field”, $G \rightarrow 1/\Phi(x)$ because that is the spirit of BD theory modification of Einstein gravity, namely the realization of “Mach’s principle”. And this BD-scalar field $\Phi(x)$ is not allowed to enter the matter sector of the full action to keep the success of “principle of equivalence” which in turn leaves it strictly massless regardless of energy scale. Besides, the kinetic term for the BD-scalar field is divided by the BD-scalar itself making the undetermined “BD ω -parameter” dimensionless. In addition, the “non-minimal” coupling of BD-scalar field to the tensor, ΦR (here R denotes the curvature scalar) looks good enough to lead one to suspect that it could play the central role in making the gravity action eventually Weyl invariant similarly to the famous non-minimal coupling term, $\xi R\phi^2$ ($\xi = \frac{1}{6}$ in 4-dim.) in conformally-invariant scalar matter field theory. Therefore, BD gravity action appears to be in a good shape for being Weyl-invariant. Now, since the BD gravity action can be viewed as a kind of action for a scalar (BD scalar) field theory coupled non-minimally to gravity (the metric tensor) like this, to investigate the possibility of its being conformally-invariant let us first briefly review some of well-known features regarding the “conformally-invariant field theories” (essentially matter field theories).

Consider a field theory of a matter field $\psi_{\mu\nu\dots}$ (which may be a spinor or tensor field) coupled Weyl invariantly to the gravitational field, namely the theory of which the classical action is invariant under the Weyl rescaling of the metric and matter fields,

$$\{x^\mu, g_{\mu\nu}(x), \psi_{\mu\nu\dots}(x)\} \longrightarrow \{x^\mu, \Omega^2(x)\tilde{g}_{\mu\nu}(x), \Omega^{-d}(x)\tilde{\psi}_{\mu\nu\dots}(x)\}. \quad (2)$$

To be more concrete, for a scalar field the scale dimension (or conformal weight) is $d = (n - 2)/2$, for a spinor (Dirac) field $d = (n - 1)/2$ and for a vector field in $n = 4$ (and only in 4) dimensions, $d = 0$. Thus the scale dimensions of the scalar and spinor fields happen to coincide with the mass (canonical) dimensions whereas it is not the case for the vector field. This is, of course, because the underlying physics of Weyl rescaling (i.e. the “local scale transformation”) is not merely a dimensional analysis although they might seem to be identical. In addition, for the Weyl invariance the action should be free of dimensionful coupling constants (including the mass). Also, as is well-known, for scalar fields one needs

to add the term $\frac{1}{2}\xi R\phi^2$ (where $\xi = (n-2)/4(n-1)$ in n -dimensions) in the action. And for spinor fields one needs to introduce a n -bein field which inherits its Weyl transformation law from the metric field, i.e., $e_\mu^a \rightarrow \Omega(x)\tilde{e}_\mu^a$.

Note that the essential requirement for the Weyl invariance is the “particular and appropriate” transformation law for the corresponding matter field (i.e., “field rescaling”), $\psi_{\mu\nu\dots}(x) = \Omega^{-d}(x)\tilde{\psi}_{\mu\nu\dots}(x)$ (where d is called “conformal weight”) that accompanies the conformal transformation of the metric, $g_{\mu\nu}(x) = \Omega^2(x)\tilde{g}_{\mu\nu}(x)$.

Here, however, it should be emphasized that these field transformation laws are not strictly implied (constrained) by the conformal transformation of metric itself, but simply “demanded” and then chosen that way for the conformal invariance. Now going back to our main objective, in order to check the possibility of the conformal invariance of BD gravity action, it appears necessary to view the BD gravity action as the action of a scalar (BD scalar) field theory coupled to gravity and then try to determine the field transformation law for the BD scalar field that is required to accompany the conformal transformation of metric for the conformal invariance, to start with. Since the scale dimension of the BD scalar field would be $d = n/2$ because the naive mass dimension of BD scalar field is $d_0[\Phi] = d_0[M_{pl}^2] = 2$ in $n = 4$ -dim., one can readily identify its field transformation law with $\Phi = \Omega^{-d}(x)\tilde{\Phi} = \Omega^{-2}(x)\tilde{\Phi}$ that accompanies the conformal transformation of the metric, $g_{\mu\nu} = \Omega^2(x)\tilde{g}_{\mu\nu}$. In fact, however, we have been misled thus far. Actually, attentive readers already must have noticed that the BD gravity action can never be conformally-invariant. The pitfall is that despite its seemingly qualified features such as the absence of dimensionful coupling constants and the non-minimal coupling term, the relative sign between the kinetic term for the BD scalar field and the non-minimal coupling term in the BD gravity action has a “wrong sign” compared to the usual relative sign between the kinetic term and the conformal coupling term in the action for a conformally-invariant scalar matter field. To illustrate this point, let us write down the actions for above two cases in *Lorentzian* signature with sign convention, $g_{\mu\nu} = \text{diag}(-+++)$ and $R_{\beta\mu\nu}^\alpha = \partial_\mu\Gamma_{\beta\nu}^\alpha - \partial_\nu\Gamma_{\beta\mu}^\alpha + \Gamma_{\mu\lambda}^\alpha\Gamma_{\beta\nu}^\lambda - \Gamma_{\nu\lambda}^\alpha\Gamma_{\beta\mu}^\lambda$,

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{g} [\Phi R - \omega g^{\mu\nu} \frac{(\partial_\mu \Phi)(\partial_\nu \Phi)}{\Phi}],$$

$$S_M = - \int d^4x \sqrt{g} \frac{1}{2} [g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) + \frac{1}{6} R \phi^2]$$

where S_{BD} is the BD gravity action and S_M is the action for a conformally-invariant scalar matter field. Pay special attention that Brans and Dicke determined the sign of BD scalar field kinetic term in Lorentzian signature such that with “positive ω ”, the contribution to the inertial reaction (i.e. the variation of Newton’s constant, $G = 1/\Phi(x)$) from the nearby matter is positive¹, as it should be. In addition, there is, indeed, another point of even more fundamental physical importance regarding the sign of BD scalar field kinetic term from the field theory’s point of view. That is, since the Euclidean action represents the energy of the system, the sign of the BD scalar field kinetic term, or equivalently, the sign of ω in the Euclidean BD gravity action given in eq.(1) should be positive. If its sign were the other way around, it would mean the negative kinetic energy of BD scalar field which implies that the BD scalar field is a “ghost field” with negative norm and thus the BD gravity Lagrangian containing it violates unitarity even at the tree level. The violation of unitarity of the BD gravity theory is apparently even more damaging than its lack of conformal invariance. (Einstein gravity does satisfy unitarity at the tree level³.) Therefore, the relative sign between the BD scalar field kinetic term and the non-minimal coupling term really has been determined correctly but it happens to be “opposite” to the relative sign between the kinetic term and the conformal coupling term for the theory of a conformally invariant scalar matter field. As a result, the BD gravity action simply cannot be invariant by its nature under the conformal transformation. And actually, one can explicitly show this conformal non-invariance of the BD gravity action as follows. Obviously, the most straightforward way of checking the conformal invariance of BD gravity action is by carrying out the conformal transformation of BD gravity action and seeing if it can be conformally-invariant under certain acceptable conditions. Thus, it can be shown that under the conformal transformations

$$g_{\mu\nu} = \Omega^2(x) \tilde{g}_{\mu\nu}, \quad \Phi = \Omega^{-2}(x) \tilde{\Phi} \quad (3)$$

the Lorentzian BD gravity action transforms as

$$S[g, \Phi] \longrightarrow S[\Omega, \tilde{g}, \tilde{\Phi}] = \frac{1}{16\pi} \int d^4x \sqrt{\tilde{g}} [\tilde{\Phi} \tilde{R} - \omega \tilde{g}^{\mu\nu} \frac{(\partial_\mu \tilde{\Phi})(\partial_\nu \tilde{\Phi})}{\tilde{\Phi}} - 2(2\omega + 3) \{ \tilde{\Phi} \Omega^{-2} \tilde{g}^{\mu\nu} (\partial_\mu \Omega)(\partial_\nu \Omega) - \Omega^{-1} \tilde{g}^{\mu\nu} (\partial_\mu \Omega)(\partial_\nu \tilde{\Phi}) \}]. \quad (4)$$

Therefore again, one can notice that BD gravity action cannot be invariant under the conformal transformations given in eq.(3) unless the BD ω -parameter can be chosen to be $\omega = -3/2$ which is unacceptable. However, an additional information one can extract out of this direct conformal transformation of BD gravity action is that although it is not invariant under the local scale transformations ($\Omega = \Omega(x)$), it can be invariant under the global scale transformations ($\Omega = \text{constant}$). Again, our general conclusion is that BD gravity is not a conformally-invariant (or local Weyl invariant) theory. Now, recall that the Einstein gravity action itself is never conformally-invariant (not even global scale invariant) either. Therefore, the conformal non-invariance of both Einstein and BD gravity actions leads us to suspect that generally the gravity action does not possess the conformal invariance.

One might object our argument thus far by claiming that setting the BD scalar field as $\Phi = \phi^2$, the BD gravity action given above now takes the form

$$S = \frac{1}{16\pi} \int d^4x \sqrt{g} [\phi^2 R - 4\omega g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi)]$$

which looks manifestly conformal invariant for suitable choice of ω . In order for this expression for the BD gravity action to be claimed to be conformally invariant, it is required to choose $\omega = -3/2$ as is obvious when compared with the conformally-coupled scalar theory case. Again this is unacceptable since it demands a negative value of ω which, as emphasized, results in the negative-definite kinetic energy and hence the violation of unitarity in the BD scalar sector of the action. (This viewpoint of ours might not agree with that of reference². This, however, is no contradiction since in ref. 2 they did not care much about the implication of the sign of ω and simply allowed ω to have any sign.)

III. Exhibition of better short-distance behavior of BD gravity theory

According to the result of our preceding study, the BD gravity theory has no conformal

invariance by its nature. Thus under the conformal transformation of metric, the field transformation law for BD scalar field needs not to be restricted by the conformal invariance condition and thus one has a full freedom in choosing the transformation law for the BD scalar field. In order to convince ourselves of this point, it might be wise to remind the definition of conformal transformation (or in a more appropriate term, Weyl rescaling). A conformal transformation is the one that scales the metric by a spacetime position-dependent scalar factor. And the theory is said to be conformally invariant if, when the metric is transformed in this manner, there exists a set of transformations on the remaining fields of the theory such that the action remains unchanged. And the term, Weyl rescaling is usually used to indicate this simultaneous action of the metric scaling and the field transformations. When a theory is not conformally invariant, however, there is no preferred choice for the field transformation accompanying the metric scaling. Then one is free to choose the transformation law for the non-metric fields. Now with these observations in mind in what follows we would like to show explicitly that the BD theory appears to have a better short-distance behavior than Einstein gravity as far as the Euclidean path integral formulation for quantum gravity is concerned. Again, consider the *Euclidean* BD-gravity action including the cosmological constant term but dropping the irrelevant surface term in eq.(1)

$$I_E[g, \Phi] = -\frac{1}{16\pi} \left\{ \int_M d^4x \sqrt{g} [\Phi(R - 2\Lambda) - \omega g^{\mu\nu} \frac{(\partial_\mu \Phi)(\partial_\nu \Phi)}{\Phi}] \right\}. \quad (5)$$

Now, if one reminds the essential point of the unboundedness of Einstein gravity action, from the above expression for Euclidean BD gravity action one can readily notice the possibility of its apparent boundedness from below. In other words, first we note that the Euclidean gravity action represents gravitational energy. Next, ignoring the surface term, in above expression for Euclidean BD gravity action, the term proportional to the curvature scalar (non-minimal coupling term) which essentially represents the kinetic energy of the tensor field can become arbitrarily negative because the curvature, in general, can become arbitrarily large and positive. Actually, this leads to the arbitrarily negative gravitational energy in general relativity or equivalently the unboundedness of Euclidean Einstein gravity

action which makes the Euclidean path integral for Einstein gravity essentially divergent. In the framework of BD gravity, however, one more gravitational degree of freedom, namely the BD scalar field Φ representing the variable Newton's constant has been added to the theory. As a result, the kinetic term for the BD scalar field in Euclidean BD gravity action contributes positive energy (as it should just like the kinetic term of ordinary scalar matter field because otherwise it would violate the unitarity as mentioned earlier) to the total gravitational energy signaling the possibility that this positive contribution from the scalar part might cancel or even overwhelm the arbitrarily negative contribution from the tensor part of gravitational degrees of freedom. Therefore, in what follows, by employing the conformal transformation scheme discussed in the preceding section, we shall show that the Euclidean path integral for BD gravity exhibits milder divergence which may be interpreted as indicating better short-distance behavior of the BD theory. First, let us examine the behavior of Euclidean BD gravity action under the conformal transformation

$$g_{\mu\nu} = \Omega^2(x) \tilde{g}_{\mu\nu}. \quad (6)$$

Since BD gravity action can never be conformally-invariant in the first place, the choice of the accompanying field transformation law for the BD scalar field does not necessarily have to be restricted by the conformal invariance condition. In other words, one may now choose the accompanying transformation law for the BD scalar field such that it transforms in an arbitrary (general) conventional *linear* way

$$\Phi = f(\Omega) \tilde{\Phi}. \quad (7)$$

Thus, under this Weyl rescaling, Euclidean BD gravity action, in eq.(5) transforms as

$$\begin{aligned} I_E[\Omega, \tilde{g}, \tilde{\Phi}] = & -\frac{1}{16\pi} \int d^4x \sqrt{\tilde{g}} f(\Omega) [\Omega^2 \tilde{\Phi} \tilde{R} - \Omega^4 \tilde{\Phi} 2\Lambda - \Omega^2 \omega \frac{(\partial_\mu \tilde{\Phi})(\partial^\mu \tilde{\Phi})}{\tilde{\Phi}} \\ & + 6\{\tilde{\Phi}(1 + \Omega \frac{f'(\Omega)}{f(\Omega)})(\partial_\mu \Omega)(\partial^\mu \Omega) + \Omega(\partial_\mu \Omega)(\partial^\mu \tilde{\Phi})\} \\ & - \omega\{\tilde{\Phi}(\Omega \frac{f'(\Omega)}{f(\Omega)})^2(\partial_\mu \Omega)(\partial^\mu \Omega) + 2\Omega(\Omega \frac{f'(\Omega)}{f(\Omega)})(\partial_\mu \Omega)(\partial^\mu \tilde{\Phi})\}] \end{aligned} \quad (8)$$

where $f'(\Omega)$ denotes the derivative of $f(\Omega)$ with respect to Ω . Therefore, even at this stage, it appears that even if one chooses the “rapidly varying” conformal factor $\Omega(x)$, there would be under a certain circumstance no problem of “negative indefiniteness” of Euclidean gravity action or “non-convergence” of Euclidean path integral for gravity in the framework of BD theory of gravity because the fluctuations of the conformal factor arising from the BD scalar field kinetic term in the third line of eq.(8) may well overwhelm or at least cancel out those arising from the kinetic term for tensor (i.e. the curvature scalar term, R) in the second line. To be more explicit, let us cast the above conformal transformed Euclidean BD gravity action to the following form

$$\begin{aligned}
I_E[\Omega, \tilde{g}, \tilde{\Phi}] = & \frac{1}{16\pi} \int d^4x \sqrt{\tilde{g}} f(\Omega) [-\Omega^2 \tilde{\Phi} \tilde{R} + \Omega^4 \tilde{\Phi} 2\Lambda + \Omega^2 \omega \frac{(\tilde{\nabla} \tilde{\Phi})^2}{\tilde{\Phi}} \\
& + \tilde{\Phi} \{ \omega X^2(\Omega) - 6(1 + X(\Omega)) \} (\tilde{\nabla} \Omega)^2 \\
& + 2\Omega \{ \omega X(\Omega) - 3 \} (\tilde{\nabla} \Omega) \cdot (\tilde{\nabla} \tilde{\Phi})]
\end{aligned} \tag{9}$$

where $X(\Omega) \equiv [\Omega \frac{f'(\Omega)}{f(\Omega)}]$. Now, since the choice of conformal factor $\Omega(x)$ and the field transformation law for the BD-scalar, $\Phi = f(\Omega) \tilde{\Phi}$ are arbitrary, if one chooses the “rapidly varying” conformal factor $\Omega(x)$ and then the field transformation law $f(\Omega)$ such that $\omega X^2(\Omega) - 6(1 + X(\Omega)) > 0$ and that generally $(\tilde{\nabla} \Omega)^2$ -term dominates over the $(\tilde{\nabla} \Omega)$ -term, then the Euclidean BD gravity action can be made as arbitrarily positive as desired. Namely, for rapidly monotonously increasing, decreasing and rapidly oscillating $\Omega(x)$, choose $f(\Omega)$ such that

$$\omega X^2 - 6(1 + X) > |\omega X - 3| > 0 \tag{10}$$

then it will do the job. For instance, choose $f(\Omega) = \Omega^{-n}$ then $\omega X^2 - 6(1 + X) = \omega n^2 + 6(n - 1)$ and $|\omega X - 3| = \omega n + 3$, and the condition in eq.(10) can always be satisfied by taking n to be $n =$ positive integer > 1 , regardless of the value of ω (which, of course, is positive)! *Note that this choice obviously includes the case when $\Phi(x)$ field follows the usual scale transformation law, $\Phi = \Omega^{-2} \tilde{\Phi}$.* Therefore, as we can see in the above example, one can always choose the conformal factor $\Omega(x)$ and the field transformation law for the BD-scalar $f(\Omega)$ such

that the Euclidean BD gravity action can be made as large and positive as wanted, even *regardless of the value of ω -parameter for a large class of the field transformation function $f(\Omega)$* . Then we may as a consequence conclude that the corresponding Euclidean path integral formulation for BD gravity can be well-defined. To see if this can be indeed the case, we refer to the argument by Gibbons, Hawking and Perry⁴ ; let G be the space of all compact, 4-metrics on a 4-dimensional manifold M . Now divide this space G into “conformal equivalent classes of metrics” under the conformal transformations, $g_{\mu\nu}(x) = \Omega^2(x)\tilde{g}_{\mu\nu}(x)$ (by conformal equivalent classes of metrics, it means that any two metrics belonging to the same conformal equivalent class are related to each other by a conformal transformation.). Then in performing the Euclidean path integral for pure BD gravity

$$Z = \int [d\tilde{g}_{\mu\nu}][d\tilde{\Phi}][d\Omega] e^{-I_E[\tilde{g}, \tilde{\Phi}, \Omega]}, \quad (11)$$

in each conformal equivalent class, first integrate over the conformal factors $\Omega(x)$ which is convergent as we have seen in the above. Next, the remaining integral over different conformal equivalent classes of metrics $\tilde{g}_{\mu\nu}$ would be convergent due to the “positive action theorem” by Schoen and Yau⁵ and so would the integral over the conformal transformed BD scalar field $\tilde{\Phi}(x)$.

IV. Discussions

To conclude, by employing the conformal transformation scheme, we showed that for a large class of the conformal transformations of BD scalar field, the Euclidean BD gravity action can be shown to be bounded from below. (Of course this analysis of ours is by no means a sound proof that the Euclidean BD gravity action is indeed bounded below.) This is in contrast to the case of Einstein gravity in which the same conformal transformation scheme demonstrates that its Euclidean action is unbounded below and hence its Euclidean path integral formulation is ill-defined. Therefore this comparison between the two theories appears to expose the fact that the BD gravity theory has a better short-distance behavior since the typical quantities involved in quantum theory exhibit milder divergences. And the BD scalar field which represents the variable Newton’s constant with space and time

embodying the spirits of Mach's principle and Dirac's large number hypothesis plays the central role in making the theory better behaved at short-distance scales.

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